ISRAEL JOURNAL OF MATHEMATICS, Vol. 42, No. 4, 1982

A COMBINATORIAL CONDITION FOR THE EXISTENCE OF POLYHEDRAL 2-MANIFOLDS

BY

U. BETKE AND P. GRITZMANN

ABSTRACT

Let $\mathcal{P}$ denote a polyhedral 2-manifold, i.e. a 2-dimensional cell-complex in $\mathbf{R}^d$ ($d \geqq 3$) having convex facets, such that set($\mathcal{P}$) is homeomorphic to a closed 2-dimensional manifold. Let E be any subset of odd valent vertices of $\mathcal{P}$, and c_E its cardinality. Then for the number $c_{P(E)}$ of facets containing a vertex of E the inequality $2c_{P(E)} \geqq c_E + 1$ is proved. This local combinatorial condition shows that several combinatorially possible types of polyhedral 2-manifolds cannot exist.

A polyhedral 2-manifold $\mathcal{P}$ is a 2-dimensional cell-complex in $\mathbf{R}^d$ ($d \geqq 3$), whose facets are convex polygons, such that set($\mathcal{P}$) is homeomorphic to a closed 2-dimensional manifold.

Given an abstract 2-dimensional cell-complex which has the structure of a combinatorial 2-manifold, the question arises, whether there exists a polyhedral 2-manifold which is combinatorially equivalent to it.

For a cell-complex $\mathcal{P}$ let $F_0(\mathcal{P})$ denote the set of all vertices of $\mathcal{P}$. For every vertex e of $\mathcal{P}$ the valence val(e, $\mathcal{P}$) is the number of polygons of $\mathcal{P}$ containing e.

It is easy to see that simple polyhedral 2-manifolds, i.e. where all the vertices are 3-valent, do not exist apart from the case of genus 0.

In [1] it is shown that for orientable polyhedral 2-manifolds $\mathcal{P}$ the "valence-distance" $\Sigma_{e \in F_0(\mathcal{P})}(\mathrm{val}(e, \mathcal{P}) - 3)$ is bounded from below by a constant determined by the genus of set($\mathcal{P}$). Here we give a local combinatorial condition for the existence of polyhedral 2-manifolds.

THEOREM. *Let $\mathcal{P}$ be a polyhedral 2-manifold, E any subset of odd valent vertices of $\mathcal{P}$, and $c_E := \mathrm{card}(E)$ its cardinality. Then for the number $c_{P(E)}$ of facets containing a vertex of E we have*:

$$2c_{P(E)} \geqq c_E + 1.$$

Received May 4, 1981 and in revised form January 7, 1982

REMARK. In locally changing the simple combinatorial 2-manifolds with minimal number of facets of genus g greater than one (cutting off a vertex and suitably inserting other complexes) one obtains non-trivial examples of combinatorial 2-manifolds which by this theorem cannot exist as polyhedral 2-manifolds. The above condition further excludes special types of equivelar polyhedral 2-manifolds (cf. [2]). The series of polyhedral 2-manifolds established in [3] shows that a similar condition involving even valent vertices is not valid, so that the restriction above is natural.

PROOF. For $e \in E$ and a polygon p of $\mathcal{P}$ containing e let $\beta(e,p)$ denote the internal angle of p at e. We set $\gamma(e) := \Sigma\, \beta(e,p)$ where the sum is taken over all polygons p containing e.

Since the star $\mathrm{st}(e,\mathcal{P})$ is topologically a disc, we can list the facets of $\mathrm{st}(e,\mathcal{P})$ in cyclic order. Let $p_1,\cdots,p_{\mathrm{val}(e,\mathcal{P})}$ be such an enumeration and let, for $i = 1,3,\cdots,\mathrm{val}(e,\mathcal{P})-2$, α_i denote the smaller of the two angles determined by the two edges of p_i and p_{i+1} containing e and belonging to exactly one of the two polygons. Then clearly

$$\alpha_i \leqq 2\pi - \beta(e,p_i) - \beta(e,p_{i+1})$$

and equality holds if and only if the affine hulls $\mathrm{aff}(p_i)$ and $\mathrm{aff}(p_{i+1})$ are the same. Since

$$\beta(e, p_{\mathrm{val}(e,\mathcal{P})}) \leqq \alpha_1 + \alpha_3 + \cdots + \alpha_{\mathrm{val}(e,\mathcal{P})-2}$$

we have

$$\beta(e, p_{\mathrm{val}(e,\mathcal{P})}) \leqq \sum_{i=1}^{\mathrm{val}(e,\mathcal{P})-1} (\pi - \beta(e,p_i)),$$

where equality holds if and only if $\mathrm{val}(e,\mathcal{P}) = 3$ and $\mathrm{aff}(p_1) = \mathrm{aff}(p_2) = \mathrm{aff}(p_3)$. Thus

$$\gamma(e) \leqq (\mathrm{val}(e,\mathcal{P}) - 1)\pi$$

and therefore

$$\sum_{e \in E} \gamma(e) \leqq \pi \sum_{e\in E} \mathrm{val}(e,\mathcal{P}) - \pi c_E, \tag{1}$$

with the above condition of equality. Because of the elementary identity

$$\sum_{e \in F_0(p)} \beta(e,p) = \pi(\mathrm{card}(F_0(p)) - 2)$$

for a polygon p and since $\beta(e,p) < \pi$, we have

$$\sum_{e \in F_0(p) \cap E} \beta(e,p) \geqq \pi(\text{card}(F_0(p) \cap E) - 2),$$

which yields

$$\sum_{e \in E} \gamma(e) \geqq \pi \sum_{e \in E} \text{val}(e, \mathcal{P}) - 2\pi c_{P(E)}, \tag{2}$$

where equality holds if and only if $F_0(p) \subset E$ for any polygon p considered above.

From (1) and (2) we get

$$\pi \sum_{e \in E} \text{val}(e, \mathcal{P}) - 2\pi c_{P(E)} \leqq \sum_{e \in E} \gamma(e) \leqq \pi \sum_{e \in E} \text{val}(e, \mathcal{P}) - \pi c_E$$

and thus

$$2c_{P(E)} \geqq c_E. \tag{3}$$

Since equality cannot occur in (1) and (2) simultaneously (3) is a strict inequality which completes the proof of the theorem.

References

1. P. Gritzmann, *Upper and lower bounds of the valence functional*, to appear.
2. P. McMullen, Chr. Schulz and J. M. Wills, *Equivelar polyhedral manifolds in E^3*, to appear.
3. P. McMullen, Chr. Schulz and J. M. Wills, *Polyhedral manifolds in E^3 with unusually large genus*, in preparation.

Universität Siegen
Hölderlinstr. 3
D-5900 Siegen 21, FRG